ON SUFFICIENT CONDITIONS FOR THE ABSENCE OF PERIODIC TRAJECTORIES IN CONSERVATIVE SYSTEMS

(O DOSTATOCHNYKH USLOVIAKH OTSUTSTVIIA PERIODICHESKIKH TRAEKTORII DLIA KONSERVATIVNYKH SISTEM)

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The problem of determination of closed trajectories, and of the limiting cycles of mechanical systems presents great mathematical difficulties, because not only the local but also the general properties of the trajectories must be studied.

There are no general rules for finding periodic trajectories, but certain necessary conditions for the existence of such trajectories can be obtained from the Poincaré theory of indices [1]. The well known Whittaker [2] and Bendixon [3] criteria permit us to to establish the regions in which the periodic trajectories can exist, application of these criteria is however difficult.

We are presenting here the conditions under which the existence of such trajectories is excluded.

Such conditions for autonomous systems, the so called 'negative criteria' [4], were found by Poincaré [1], Bendixon [3], and Dulac, [5]. Various generalizations of these criteria for autonomous systems are presented in [6 and 7].

The purpose of this note is to show the existence of a 'negative criterion' for conservative systems as well.

1. Let a point M of a mechanical system move in a conservative force field its potential being $V(x, y)$, with prescribed constant energy h. We shall write the differential equations of its trajectory, in the form [8]

$$y'' = (1 + y'^2)\left(-\frac{\partial\Phi}{\partial x}y' + \frac{\partial\Phi}{\partial y}\right) \qquad \left(\Phi = \ln\sqrt{2(h - V(x, y))}\right) \tag{1.1}$$

Introducing the angle $\psi(x, y) = \tan^{-1} y'$ between the velocity vector v and the positive direction of the x-axis, we obtain the differential relation

$$d\psi(x, y) = \frac{\partial\Phi}{\partial y}\,dx - \frac{\partial\Phi}{\partial x}\,dy \tag{1.2}$$

The behavior of the trajectory of our system will depend essentially on the distribution and character of the singular points O_j of the function $\Phi(x, y)$. Let us introduce the concept of a 'quasi-index' of the singular point O_j as the limiting value of the integral

$$J_j = \frac{1}{2\pi} \lim_{r \to 0} \oint_{(\gamma_j)} \frac{\partial \Phi}{\partial y}\, dx - \frac{\partial \Phi}{\partial x}\, dy \tag{1.3}$$

where the integration is performed along the contour (γ_j) of a circle of small radius r, with its center at O_j. For the ordinary point the quasi-index will be equal to zero, and it differs from the Poincaré index in, that the latter must be an integer, whereas the quasi-index of a singular point can be any real number.

Consider for example a force fiel generated by the centers of attraction O_j with potentials $V_j = -A/r_j^n$. It can be easily shown that the quasi-index of a singular point O_j equals $J_j = \frac{1}{2}n$.

2. Let us consider a region (D) where the function $\Phi(x, y)$ has isolated singular points O_j $(j = 1, 2, \ldots, k)$, while at the remaining points it is continuous, and has continuous partial derivatives of the first and second order.

Let our system undergo a periodic motion while remaining in the phase plane in the region (D). Then, there exists in (D) a closed (without intersections) contour (C) along which the integral of the right-hand side of the Pfaff form (1.2), is

$$\oint_{(C)} \frac{\partial \Phi}{\partial y}\, dx - \frac{\partial \Phi}{\partial x}\, dy = 2\pi \tag{2.1}$$

Let all the singular points O_j $(j = 1, 2, \ldots, k)$ of the function $\Phi(x, y)$ which are in (D) be inside the contour (C).

Let us change the line integral in the left-hand side of (2.1) into an area integral. To do this, we shall separate, inside the orbit (C), the regions bounded by circles (γ_j) of small radii r, their centers coinciding with the singular points O_j. Considering the multiply connected region (σ^*) bounded by a complex contour $(\Gamma) = (C) + (\gamma_1) + (\gamma_2) + \ldots + (\gamma_k)$, and using Green's theorem, we obtain

$$\oint_{(\Gamma)} \frac{\partial \Phi}{\partial y}\, dx - \frac{\partial \Phi}{\partial x}\, dy = -\iint_{(\sigma^*)} \Delta \Phi\, dx\, dy \tag{2.2}$$

where (σ^*) is the region bounded by the contour (Γ).

Letting now the radii r_k of the circles (γ_k) in the formula (2.2) to decrease and passing to the limit as $r_k \to 0$, we obtain, by (2.1) and (1.3)

$$-\frac{1}{2\pi} \iint_{(\sigma)} \Delta \Phi\, dx\, dy = 1 - J \qquad (J = J_1 + \ldots + J_k) \tag{2.3}$$

Here (σ) is the region bounded by the contour (C) and J is the sum of quasi-indices of the singular points O_j which are inside the contour (C).

Theorem. Suppose that in our region the sum of quasi-indices J satisfies one of the following conditions

$$(a) \quad -\infty < J < 1, \qquad (b) \quad J = 1, \qquad (c) \quad 1 < J < +\infty \tag{2.4}$$

If, at the same time the function $\Delta\Omega$ either has a constant sign or is equal to zero in the region (D), then we have the following sign relations corresponding to the cases given in (2.4)

$$(a) \quad \triangle\Phi \geqslant 0, \qquad (b) \quad \triangle\Phi > 0 \; (\triangle\Phi < 0), \qquad c) \quad \triangle\Phi \leqslant 0 \tag{2.5}$$

and this is a sufficient condition for the absence of closed trajectories in our region (D). Consequently, $\Phi(x, y)$ should belong in the case (a) to the class of subharmonic functions, in the case (c) to the class of superharmonic functions, while the case (b) can be expressed by the single condition $\Delta\Phi \neq 0$.

On the strength of (2.3) and (2.4) the sufficiency of the established criteria (2.5) is obvious.

3. Let us consider some examples which show, that existence of periodic trajectories in the regions, where the negative criteria (2.5) are not satisfied, is possible.

(1) In the force field with the logarithmic potential $V = A \ln r$ point M can undergo periodic motions, moving around a circle (C) of an arbitrary radius r with its center at the origin of coordinates. Here

$$\Phi_x = \frac{-Ax}{2r^2(h - A\ln r)}, \qquad \Phi_y = \frac{-Ay}{2r^2(h - A\ln r)}, \qquad (\Phi = \ln \sqrt{2(h - A\ln r)})$$

hence the quasi-index of the singular point $(r = 0)$ will be equal to

$$J = \frac{1}{2\pi}\lim_{r\to 0}\oint_{(\gamma)} \frac{A(x\,dy - y\,dx)}{2r^2(h - A\ln r)} = \lim_{r\to 0}\frac{A}{2(h - A\ln r)} = 0$$

that is, we are considering the case (a). On the other hand the condition of the negative criterion $\Delta\Phi \geqslant 0$ is satisfied, because in our case

$$\triangle\Phi = \frac{-A^2}{2r^2(h - A\ln r)^2} < 0$$

(2) As a second example let us consider the motion of a point M in a central force field with the potential $V = -A / r^n$ $(A, n > 0)$.

From direct calculations we obtain

$$\triangle\Phi = \frac{Ahn^2r^{n-2}}{2(A + hr^n)^2} \qquad (\Phi = \ln\sqrt{2(h + A/r^n)}) \tag{3.1}$$

consequently, when $h = 0$ we have $\Delta\Phi = 0$, while when $h \neq 0$ the sign of $\Delta\Phi$ is the same as the sign of the constant energy h, that is sign $(\Delta\Phi)$ = sgn (h). We shall consider now a periodic motion of the point M on a circle (C) of radius r, its center at the origin. From physical considerations it follows, that on the contour (C) the condition $v^2 = An/r^n$ should be satisfied, and consequently on the strength of the energy integral we get

$$\frac{A(n-2)}{2r^n} = h \tag{3.2}$$

that is the sign of h will depend on the magnitude of the exponent n characterizing the potential of the force field. If we write the expansion of Φ_x and Φ_y in the neighborhood of zero $(r = 0)$

$$\Phi_x = -\frac{nx}{r^2} + \ldots, \qquad \Phi_y = -\frac{ny}{r^2} + \ldots$$

we can calculate the quasi-index of the singular point ($r = 0$)

$$J = \frac{1}{2\pi} \lim_{r\to 0} \oint_{(\gamma)} \frac{n\,(x\,dy - y\,dx)}{2r^2} = \frac{n}{2} \tag{3.3}$$

For different values of the exponent n we have by (3.1), (3.2), and (3.3)

1) $n < 2, \quad J < 1, \quad h < 0, \quad \triangle\Phi < 0$
2) $n = 2, \quad J = 1, \quad h = 0, \quad \triangle\Phi = 0$
3) $n > 2, \quad J > 1, \quad h > 0, \quad \triangle\Phi > 0$

Here on the strength of (2.5) the negative criteria are not satisfied, and periodic trajectories exist. This shows the essence of the negative criteria (2.5).

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